

## MAC-CPTM Situations Project

### *Situation 21: Exponential Rules*

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#### **Prompt**

In an Algebra II class, the teacher wrote the following problem on the board:  
 $x^m \cdot x^n = x^5$ . The students had just finished reviewing the rules for exponents. The teacher asked the students to make a list of values for  $m$  and  $n$  that made the statement true. After a few minutes, one student asked, “Can we write them all down? I keep thinking of more.”

#### **Commentary**

At first glance, this Situation seems simply to be about rules for exponents. And certainly the rule  $x^m \cdot x^n = x^{m+n}$  is applicable here, and is discussed in Focus 2. But looking a little deeper we find issues of domain that must be addressed before applying any rules. For example, the values of  $m$  and  $n$  are limited, and their domain depends on the value of  $x$ . Conversely, depending on what the values of  $m$  and  $n$  are, the domain of  $x$  is limited. This is the theme of Focus 1.

## Mathematical Foci

### *Mathematical Focus 1*

For  $x^m$  and  $x^n$ , the value of  $x$  determines the domain for  $m$  and  $n$ . Specifically if  $x$ ,  $m$ , and  $n$  are real numbers, then:

- If  $x = 0$ , then  $m > 0$  and  $n > 0$ . This is because 0 raised to a negative power (ex.  $0^{-2}$ ) is undefined.  $0^0$  is unique as well. In most cases, it is considered to be undefined. However, there are instances (for example, in the Binomial Theorem) in which  $0^0$  is defined as 1.
- If  $x > 0$ , then  $m$  and  $n$  could be any real numbers. This is what is assumed in the rule  $x^m \cdot x^n = x^{m+n}$  and is discussed in Focus 2.
- If  $x < 0$ , then  $m$  and  $n$  must be integers or rational numbers whose denominator (when the number is in lowest terms) is odd.

Here are some examples that may clarify this last point. Let  $x = -2$ . With this (or any negative number) as a base, the exponent,  $m$ , may be any integer and the result will be a real number:

$m$	$(-2)^m$
-2	0.25
-1	-0.5
0	1
1	-2
2	4
3	-8

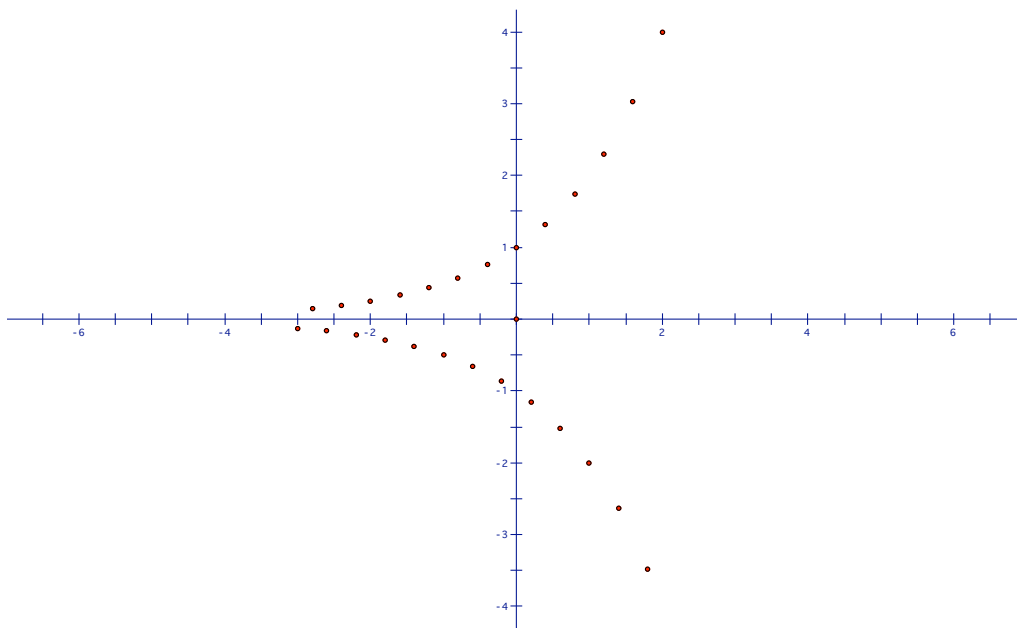
The exponent,  $m$ , may also be a rational number as long as its denominator (when the number is reduced to lowest terms) is odd. This is because if the exponent is a fraction with an even denominator (such as  $\frac{1}{2}$ ), the exponent indicates an even root. An even root of a negative number does not give a real result. If we restrict the domains of  $x$ ,  $m$ , and  $n$  to the real numbers, then an even root of a negative number lies outside the domain of  $x$ ,  $m$ , and  $n$ . If we were not limited to the real domain,  $m$  could be any number as long as  $x \neq 0$  (as explained in a) above).

Examining a list of values of  $(-2)^m$  and the resulting graph also provides insight into these restrictions. The following graph is the result of plotting the points in the table below. These points were chosen because they show that  $(-2)^m$  is a real

number for certain values of  $m$  and not for others. Notice that the values for  $m$  that result in  $(-2)^m$  having no real solution ( $\emptyset$ ) are those with numerators that, if written as fractions in lowest terms, would have even denominators. Notice also that the real values of  $(-2)^m$  in the table alternate between positive and negative. This is because a negative base raised to an even power is positive (such as  $(-2)^{-2} = 0.25$ ), and a negative base raised to an odd power is negative (such as  $(-2)^{-3} = -0.125$ ).

$m$	$(-2)^m$	$m$	$(-2)^m$	$m$	$(-2)^m$	$m$	$(-2)^m$	$m$	$(-2)^m$
-3	-0.125	-2	0.25	-1	-0.5	0	1	1	-2
-2.9	$\emptyset$	-1.9	$\emptyset$	-0.9	$\emptyset$	0.1	$\emptyset$	1.1	$\emptyset$
-2.8	0.143...	-1.8	-0.287...	-0.8	0.574...	0.2	-1.148...	1.2	2.297...
-2.7	$\emptyset$	-1.7	$\emptyset$	-0.7	$\emptyset$	0.3	$\emptyset$	1.3	$\emptyset$
-2.6	-0.164...	-1.6	0.329...	-0.6	-0.659...	0.4	1.319...	1.4	-2.639...
-2.5	$\emptyset$	-1.5	$\emptyset$	-0.5	$\emptyset$	0.5	$\emptyset$	1.5	$\emptyset$
-2.4	0.189...	-1.4	-0.378...	-0.4	0.757...	0.6	-1.515...	1.6	3.031...
-2.3	$\emptyset$	-1.3	$\emptyset$	-0.3	$\emptyset$	0.7	$\emptyset$	1.7	$\emptyset$
-2.2	-0.217...	-1.2	0.435...	-0.2	-0.870...	0.8	1.741	1.8	-3.482...
-2.1	$\emptyset$	-1.1	$\emptyset$	-0.1	$\emptyset$	0.9	$\emptyset$	1.9	$\emptyset$

With  $m$  on the horizontal axis and  $(-2)^m$  on the vertical, the result is:



Thus far we have concentrated on how the value of the base,  $x$ , in  $x^m$  determines the domain of the exponent,  $m$ . But it is also true that if we want  $x^m$  to be defined, then the value of  $m$  puts certain limitations on  $x$ .

- d) If  $m$  is a positive integer, then  $x$  may be any real number.
- e) If  $m$  is a non-zero rational number with an odd denominator (when written in lowest terms), then  $x$  may be any real number.
- f) If  $m$  is an integer  $\leq 0$ , then  $x \neq 0$ . This is the contrapositive of a) above.

- g) If  $m$  is an irrational number or a nonzero rational number with an even denominator (when written in lowest terms), then  $x \geq 0$ . (See previous discussion of even denominators.) This is the contrapositive of c) above.

*Mathematical Focus 2:*

The exponent rule  $x^m \cdot x^n = x^{m+n}$  may be applied to the equation  $x^m \cdot x^n = x^5$  if the value of  $x$  is restricted to positive numbers. That is, if  $x > 0$ , and  $m$  and  $n$  are real numbers, then  $x^m \cdot x^n = x^{m+n}$ .

In this case there are various things to consider when determining the values of  $m$  and  $n$ . If their values are restricted to natural (counting) numbers, then there are a finite number of solutions for  $(m,n)$ : (1,4), (2,3), (3,2), and (4,1). If  $m$  and  $n$  are not limited to the natural numbers, then there are infinite solutions for  $(m,n)$  since there are infinite solutions to the equation  $m + n = 5$ . This is not to say that  $m$  and  $n$  could be any numbers, only that they may be any two values whose sum is 5. (For example, they must be rational because their sum (5) is rational.)

The restriction  $x > 0$  is an important one if we are limiting ourselves to real numbers. If, for example,  $m = 1.5$ .

$$x^m = x^{1.5} = x^{3/2} = \sqrt{x^3}$$

For this result to be a real number,  $x$  cannot be negative. If the restriction to real numbers were not in effect,  $x$  could be any number.